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TRANSITION RADIATION IN AN ELASTIC WHEEL

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The phenomenon of transition radiation has been known in physics since the middle of the present century [1]; at present it has been investigated in detail in electrodynamics [2] and acoustics [3]. In connection with the development of high-speed ground transportation and the increase in machinery speed of operation in the last decade, it has become obvious that transition radiation plays a significant role in the interaction of elastic structures with moving objects [4–7]. A clear example of such a structure is the current collector unit of an electrical vehicle, where the moving collector excites elastic waves in a contact wire due to the presence of various clips, holders, airswitches, etc. [8].

The radiation generated can, on the one hand, cause some undesirable events, like an increase in the amplitude of oscillations of the collector [7] or losses of contact [9], and, on the other hand, is a natural field of deformations, making it possible to determine the state of the system. Another important example of a mechanical system where the moving load can excite elastic waves is an elastic wheel [10], a typical element of most mechanisms. If the wheel is nonuniform over the angular coordinate (i.e., if there are spokes, hanger brackets, disk brakes, etc.), transition radiation will occur. The investigation of transition radiation as applied to an elastic wheel is of both theoretical and practical interest. From the theoretical viewpoint, it is interesting to analyze the features of radiation that are related to the closedness of an elastic system. The necessity of elaborating a proper theory of aircraft gear instability ("shimmy"), adequate to modern landing speeds, shows the practical importance of the problem.

The goal of the present study is to qualitatively investigate the phenomenon of transition radiation in an isolated elastic system. We simulate the wheel as a wire expanded with springs, the stiffness of which is uniformly distributed over the angle (Fig. 1). The nonuniformity is presented in the form of concentrated, elastic-inertial "spokes" placed equidistantly along the wheel perimeter. It is assumed that, as a result of interaction with a ground surface (or any similar object), the wheel is affected by a force which is constant in modulus and is directed along the radius; the point of application of this force rotates at a constant angular velocity. Using the method of images [11], we have obtained an exact solution to the problem of steady oscillations of the wheel. It has been established that under the effect of a moving load, the transition radiation of elastic waves may occur in the wheel. The spectrum of this radiation is discrete and the phase velocity of each harmonic is equal to the load velocity. It has been shown that a resonance can take place when the wavelength of one of the harmonics is a multiple of the wheel length.

Let a nontensible wire of length l be expanded by the springs, which are uniformly distributed over the angle with density k per unit length, one end of each being tied to a solid axle. If the length a of the unstrained spring is greater than $R = 1/2\pi$, the wire will be expanded, acquiring the form of a ring. The wire tension, according to [12], is determined as $T = kR(a - R)$. We will also assume that, along with springs, the wire is linked with the axle by means of concentrated, equidistant, elastic-inertial "spokes," the number of which is N. We consider all "spokes" equal, with mass m, stiffness k_0, and viscosity ν. The length of the unstrained "spoke" element is assumed to be equal to R.

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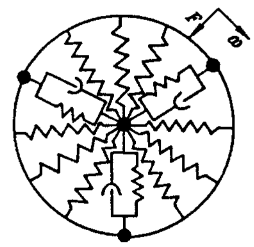

Fig. 1

Let us examine the forced oscillations of the wire excited by a radial, modulus-constant force F, with the point of application rotating at a constant angular velocity ω. According to [12, 13], the equations describing small radial wire oscillations are given by

$$\begin{gathered}\rho U_{\tau\tau} - TU_{ss} + kU = -F\delta(s - R\omega\tau + l\{\omega\tau/2\pi\}), \quad 0 \leqslant s \leqslant l, \ 0 \leqslant \tau < \infty, \quad [U]_{s=nl/N} = 0, \\ U(nl/N, \tau) = y_n(\tau), \quad T[U_s]_{s=nl/N} = m\ddot{y}_n + \nu\dot{y}_n + k_0 y_n, \quad U(s+l, \tau) = U(s, \tau), \quad 1 \leqslant n \leqslant N\end{gathered} \tag{1}$$

(in a linear approximation, this model describes tangent oscillations independent of radial ones). Here, $U(s,\tau)$ is the radial displacement of the wire; $y_n(\tau)$ is the radial displacement of nth inertial "spoke" element; τ is the time; $s = R\varphi$ is an angular variable (φ denotes angle); ρ is the wire density per unit length; $\delta(\ldots)$ is a Kronecker delta; $\{b\}$ denotes the integer part of the number b; $[f]_{s=c} = f(c+0) - f(c-0)$.

We will seek the solution to problem (1) using the method of images. Assuming the elastic system infinite, we choose auxiliary imaginary sources to satisfy the periodicity condition $U(s+l,\tau) = U(s,\tau)$. It is evident that the sources of the force F moving as $s_k = \omega R\tau + kl$ ($k = 0, \pm1, \pm2\ldots$) satisfy these requirements. Consequently, after nondimensionalization, the auxiliary problem which has a solution identical to that of system (1) for $s \in [0, l]$ will take the form (Fig. 2)

$$\begin{gathered}U_{tt} - U_{xx} + U = -P\sum_{k=-\infty}^{\infty} \delta(x - vt + Ndk), \quad -\infty < x < \infty, \ -\infty < t < \infty, \\ [U]_{x=nd} = 0, \quad U(nd, t) = y_n(t), \quad [U_x]_{x=nd} = M\ddot{y}_n + \delta\dot{y}_n + Ky_n, \quad n = 0, \ \pm1, \ \pm2, \ldots,\end{gathered} \tag{2}$$

where $x = sh/c$, $t = h\tau$ ($h = (k/\rho)^{1/2}$, $c = (T/\rho)^{1/2}$) are the dimensionless coordinate and time, respectively; $v = \omega R/c$ is the dimensionless velocity of the load (here and below $v < 1$); $d = hl/cN$ is the dimensionless distance between the "spokes"; $P = F/\rho hc$ is the load; $M = mhc/T$, $\delta = \nu c/T$, $K = k_0 c/hT$ are the dimensionless mass, viscosity, and stiffness of the "spoke"; the dot above y_n denotes differentiation with respect to the dimensionless time t.

To determine the steady-state wire oscillations, we apply to (2) the Fourier transform

$$V(x,\omega) = \int_{-\infty}^{\infty} U(x,t)\exp(i\omega t)\,dt, \qquad z_n(\omega) = \int_{-\infty}^{\infty} y_n(t)\exp(i\omega t)\,dt.$$

For the transform we obtain

$$\begin{gathered}V_{xx} + (\omega^2 - 1)V = \frac{P}{v}\exp(i\omega x/v)\sum_{k=-\infty}^{\infty}\exp(i\omega dNk/v), \\ [V]_{x=nd} = 0, \quad V(nd,\omega) = z_n(\omega), \quad [V]_{x=nd} = (K - i\delta\omega - M\omega^2)z_n.\end{gathered} \tag{3}$$

For $x \in [0, d]$ the general solution to (3) is given by

$$V(x) = A\exp(ix\sqrt{\omega^2 - 1}) + B\exp(-ix\sqrt{\omega^2 - 1}) - S\exp(i\omega x/v), \tag{4}$$

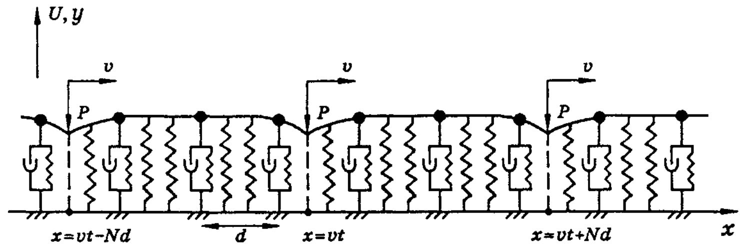

Fig. 2

where

$$S(\omega) = \frac{Pv}{\omega^2(1-v^2)+v^2} \sum_{k=-\infty}^{\infty} \exp(i\omega dNk/v).$$

As the "spokes" are equidistant and the load moves uniformly, system (2) must satisfy the periodicity condition

$$U(x,t) = U(x+d,\ t+d/v), \tag{5}$$

the transform of which is

$$V(x+d) = V(x)\exp(i\omega d/v). \tag{6}$$

Combined with the periodicity condition, expression (4), which describes $V(x)$ for $x \in [0,d]$, allows us to extend the solution of (3) to any x. In particular, to $x \in [d, 2d]$

$$V(x) = \exp(i\omega d/v)(A\exp(i(x-d)\sqrt{\omega^2-1}) + B\exp(-ix(x-d)\sqrt{\omega^2-1})) - S\exp(i\omega x/v). \tag{7}$$

Assuming that $z_0(\omega) = C(\omega)$, from the periodicity condition (6) we derive

$$z_n = C\exp(i\omega dn/v). \tag{8}$$

Joining the solutions of (4) and (7) at $x = d$ and using (8), we write the following system of equations to determine A, B, and C:

$$A+B-S=C, \quad A\exp(id\sqrt{\omega^2-1}) + B\exp(-id\sqrt{\omega^2-1}) = (A+B)\exp(i\omega d/v),$$

$$i\sqrt{\omega^2-1}(\exp(i\omega d/v)(A-B) - A\exp(id\sqrt{\omega^2-1}) + B\exp(-id\sqrt{\omega^2-1})) = C(K - i\delta\omega - M\omega^2)\exp(i\omega d/v),$$

Now, switching to the inverse transform, we obtain an exact solution to problem (2), describing the steady-state oscillations of a wire for $x \in [0,d]$:

$$U(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{S}{\Delta}(Gp(p-\gamma^-)\exp(ix\sqrt{\omega^2-1})$$

$$+ Gp(\gamma^+ - p)\exp(-ix\sqrt{\omega^2-1}) - \Delta\exp(i\omega x/v))\exp(-i\omega t)d\omega. \tag{9}$$

The wire displacement for $x \in [d, Nd]$ is derived with the help of the periodicity condition (5).

The solution to problem (2) for $x \in [0, Nd]$ is identical to that of problem (1); hence, expressions (9) and (5) taken together determine an exact solution to the starting problem (1).

The poles in (9) satisfy the equation

$$\Delta = 0 \Leftrightarrow \cos(æ d) = \cos(\omega d/v), \tag{10}$$

where $\cos(æ d) = \cos(d\sqrt{\omega^2-1}) + G\sin(d\sqrt{\omega^2-1})/2\sqrt{\omega^2-1}$ is the dispersion equation for the tense wire (string) lying on equidistant props. The roots of Eq. (10) are a discrete set of frequencies and determine the harmonics excited by the load. This equation has a simple physical interpretation. Indeed, when the load crosses a "spoke," transition radiation of elastic waves is excited in the wire [4], having continuous spectrum. Because of the periodicity of the elastic system, the radiation fields from every "spoke" come in phase only at certain frequencies. Thus, when the regime is steady, the spectrum of radiation becomes discrete and the frequencies of the harmonics are determined by Eq. (10), which is, in fact, a condition for the "resonance" of radiation fields (for this reason, the transition radiation in a periodically nonuniform media is called the resonance transition radiation [2]).

Equation (10) can be rewritten in a more compact form. Indeed, it follows from (10) that

$$\text{æ} + 2\pi j/d = \omega/v, \qquad j = 0,\ \pm 1,\ \pm 2\dots. \tag{11}$$

The expression $\text{æ} + 2\pi j/d$ is nothing else but the wave number k_j of the jth spatial harmonic in a periodically nonuniform medium [14]. Therefore, (10) can be rewritten as $v_j^{\rm ph} = v$, where $v_j^{\rm ph} = \omega/k_j$ is the phase velocity of the jth harmonic. Thus, the phase velocity of each harmonic is equal to the load velocity.

Determination of the resonance parameters of a system that yield a sudden increase in oscillation amplitude is important for practical needs. Let us find resonance conditions for our model of an elastic wheel. Without restricting the generality, we will find them by examining the oscillations of only one "spoke". Since the choice of time moment is not important when determining the resonance parameters, to shorten the calculations we choose a "spoke" having number $n = 0$, at zero time $t = 0$. In accordance with (9), the displacement of the inertial element of the chosen "spoke" at $t = 0$ is given by

$$y_0(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} z(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Delta_3/\Delta d\omega = -\frac{1}{4\pi}\int_{-\infty}^{\infty} S(\omega)\,\frac{\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,)}{\cos(\omega d/v) - \cos(\text{æ} d)}\,d\omega, \tag{12}$$

where $S(\omega)$ is the same as in (4). We perform the integration (12) using the residue theory. Splitting the integral into three parts, we write

$$y_0(0) = -(I_1 + I_2 + I_3)/4\pi,$$

where

$$I_1 = \int_{-\infty}^{\infty} D^{-1}(\omega)(\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,))\,d\omega;$$

$$I_2 = \sum_{k=1}^{\infty}\int_{-\infty}^{\infty} D^{-1}(\omega)(\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,))\exp(i\omega dNk/v)\,d\omega;$$

$$I_3 = I_2^*; \quad D(\omega) = (\omega^2(1-v^2) + v^2)(\cos(\omega d/v) - \cos(\text{æ} d))/Pv;$$

here the asterisk denotes complex conjugation

We calculate I_1, I_2, and I_3 sequentially. For I_1 we write

$$I_1 = \frac{1}{2}\Big(\int_{-\infty}^{\infty} D^{-1}(\omega)(\exp(i\omega d/v) - \exp(id\sqrt{\omega^2-1}))\,d\omega + \text{c.c.}\Big). \tag{13}$$

Closing the integration path through the upper half-plane of the complex variable ω for the first integral from (13) and through the lower half-plane for the second one, we obtain

$$I_1 = \pi i \sum_m \operatorname*{Res}_{\mathrm{Im}(\omega_m)>0}\Big\{D^{-1}(\omega_m)(\exp(i\omega_m d/v) - \exp(id\sqrt{\omega_m^2-1}\,))\Big\}$$

$$-\pi i \sum_l \operatorname*{Res}_{\mathrm{Im}(\omega_l)<0}\Big\{D^{-1}(\omega_l)(\exp(-i\omega_l d/v) - \exp(-id\sqrt{\omega_l^2-1}\,))\Big\} - \int_{-1}^{1} D^{-1}(\omega)\,\mathrm{sh}\,(d\sqrt{\omega^2-1}\,)\,d\omega, \tag{14}$$

where ω_m and ω_l are zeros of the function $D(\omega)$ in the upper and lower half-planes of ω, respectively; $\mathrm{Res}\{f(\omega_k)\}$ is the residue of the function $f(\omega)$ at ω_k.

Now we pass to calculation of I_2. Since $N \geqslant 1$, an integral taken along the half-circumference of infinite radius in $\mathrm{Im}(\omega) > 0$ tends toward zero. Hence, closing the integration path through the upper half-plane of ω, we find

$$I_2 = \sum_{k=1}^{\infty} 2\pi i \sum_m \operatorname*{Res}_{\mathrm{Im}(\omega_m)>0} \Big\{ D^{-1}(\omega_m)(\cos(\omega_m d/v) - \cos(d\sqrt{\omega_m^2 - 1}\,)) \exp(i\omega_m dNk/v) \Big\}.$$

As a consequence of $\mathrm{Im}(\omega_m) > 0$, the derived equation can be summed over all k using a formula for the sum of an infinite geometric progression, the denominator of which is $\exp(i\omega_m dNk/v)$. Having performed this operation, we find

$$I_2 = 2\pi i \sum_m \frac{\exp(i\omega_m dN/v)}{1 - \exp(i\omega_m dN/v)} \operatorname*{Res}_{\mathrm{Im}(\omega_m)>0} \Big\{ D^{-1}(\omega_m)(\cos(\omega_m d/v) - \cos(d\sqrt{\omega_m^2 - 1})) \Big\}. \qquad (15)$$

To calculate I_3, we must close the integration path through the lower complex half-plane. Being summed over all k, the expression for I_3 takes the form

$$I_3 = -2\pi i \sum_l \frac{\exp(-i\omega_l dN/v)}{1 - \exp(-i\omega_l dN/v)} \operatorname*{Res}_{\mathrm{Im}(\omega_l)<0} \Big\{ D^{-1}(\omega_l)(\cos(\omega_l d/v) - \cos(d\sqrt{\omega_l^2 - 1}\,)) \Big\}. \qquad (16)$$

As one can see from (15) and (16), there are two cases that are suspicious for the resonance (i.e., an offset of the "spoke" $y_0(0)$ tends toward infinity when its viscosity ν tends toward zero):

(a) The equation $D(\omega) = 0$ for $\nu = 0$ has a multiple real root, and the integral (12) should seemingly diverge as the Fourier transform of the function that has a multiple root on the real axis;

(b) The root of the equation $D(\omega) = 0$ for $\nu = 0$ coincides with one of the roots of the equation $1 - \exp(\pm i\omega dN/v) = 0$, so either I_2 or I_3 tend toward infinity.

Let us carefully examine the two conditions above. Case (a) implies that the group velocity $d\omega/dk_j$ of one of the radiated harmonics coincides with the load velocity v. Certainly, the expression $d\omega/dk_j = v$ is the only resonance condition [7], but only for an infinite, periodically nonuniform guide interacting with a moving load. An isolated system does not satisfy this condition. The resonance will not occur if the wavelength of the radiated harmonic is not a multiple of the ring length. Indeed, let us suppose that the equation $D(\omega) = 0$ for $\nu = 0$ has a multiple real root ω^*. In this case, the terms of expression for $y_0(0)$ all tend toward infinity. Designating their sum as S^∞, we obtain, according to (14)–(16):

$$\begin{aligned} S^\infty = {} & \pi i \operatorname*{Res}_{\omega\to\omega^*+i0} \Big\{ D^{-1}(\omega)(\exp(i\omega d/v) - \exp(id\sqrt{\omega^2-1}\,)) \Big\} \\ & - \pi i \operatorname*{Res}_{\omega\to\omega^*-i0} \Big\{ D^{-1}(\omega)(\exp(-i\omega d/v) - \exp(-id\sqrt{\omega^2-1}\,)) \Big\} \\ & + 2\pi i \frac{\exp(i\omega^* dN/v)}{1-\exp(i\omega^* dN/v)} \operatorname*{Res}_{\omega\to\omega^*+i0} \Big\{ D^{-1}(\omega)(\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,)) \Big\} \\ & - 2\pi i \frac{\exp(-i\omega^* dN/v)}{1-\exp(-i\omega^* dN/v)} \operatorname*{Res}_{\omega\to\omega^*-i0} \Big\{ D^{-1}(\omega)(\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,)) \Big\}. \end{aligned}$$

Using the expression $\operatorname*{Res}_{\omega\to\omega^*+i0}\{f(\omega)\} = \operatorname*{Res}_{\omega\to\omega^*-i0}\{f(\omega)\}$ which follows from the definition of the residue, we rewrite S^∞ in the form

$$S^\infty = 2\pi i \Big\{ 1 + \frac{\exp(i\omega^* dN/v)}{1-\exp(i\omega^* dN/v)} + \frac{\exp(-i\omega^* dN/v)}{1-\exp(-i\omega^* dN/v)} \Big\} \operatorname*{Res}_{\omega\to\omega^*+i0} \Big[D^{-1}(\omega)(\cos(\omega d/v) - \cos(d\sqrt{\omega^2-1}\,)) \Big].$$

By bringing the expression in braces to the common denominator, we easily ascertain that it is equal to zero. Consequently, both S^∞ and the "spoke" displacement $y_0(0)$ must remain finite. Thus, the condition (a) is not a condition for the resonance.

On the contrary, the case (b) does really determine the parameters of the system at which the resonance is observed. Mathematically it is obvious, since $y_0(0)$ tends toward infinity due to the growth of one of the terms in (b). Physically condition (b) is clear enough as well. Indeed, the real roots of the equation $D(\omega) = 0$

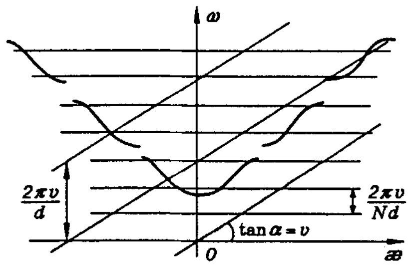

Fig. 3

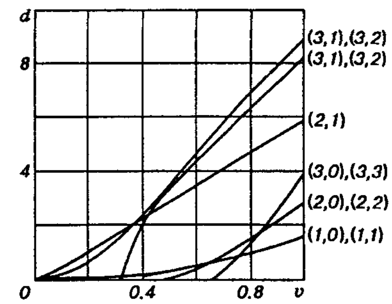

Fig. 4

are equal to those of (11), whereas the equations $1 - \exp(\pm i\omega dN/v) = 0$ have the same solutions as the equation $\sin(\omega dN/2v) = 0$. Therefore, condition (b) can be rewritten as the following system:

$$æ + \frac{2\pi j}{d} = \frac{\omega}{v}, \quad \omega = \frac{2\pi v k}{Nd}, \quad j = 0, \pm 1, \pm 2, \dots, \quad k = 0, \pm 1, \pm 2, \dots \tag{17}$$

Taking into account that $2\pi/(æ + 2\pi j/d) = 2\pi/k_j = \lambda_j$, where k_j is the wave number of the jth harmonic, and λ_j is the corresponding wavelength, from (17) we derive

$$j\lambda_j = Nd,$$

where Nd is the dimensionless length of the wheel.

Thus, the resonance condition for a wheel with "spokes" is the multiplicity of one of the radiated harmonics to the wheel length.

Figure 3 depicts a graphical solution to system (17). The dashed line represents the dispersive dependence $æ(\omega)$ for a periodically nonuniform elastic system (an unfolded wheel). The points of intersection of the family of slanted curves $\omega_j = æv + 2\pi v j/d$ to the line $æ(\omega)$ determine the frequencies of harmonics radiated under the load. Resonance will occur when ω_j coincides with one of the natural frequencies of the wheel $\omega_w = 2\pi v k/Nd$, shown in Fig. 3 by horizontal lines.

Figure 4 shows a family of curves drawn in the plane of the parameters (d, v). Resonance will occur when the parameters of the system fall on one of these curves (we put $K = 0.7$, $\nu = 0$, $M = 0.3$, $N = 2$). Of course, not all solutions of (17) are drawn here (there is a countable set of them) but only those corresponding to the three lowest natural frequencies of the wheel. There is no reason to take into account the higher frequencies because their resonances are most likely to be suppressed by the dissipation that takes place in all real wheels. We put the subscripts (j, k) at the points of intersection of the curves with the straight line $v = 1$; the first index denotes the serial number of wheel eigen frequency while the second one corresponds to the serial number of radiated harmonics. As one can see from the drawing, if the distance between the spokes d is fixed, there exists a set of load velocities at which resonance occurs. Naturally, the presence of dissipation will affect the amplitude of resonance oscillations differently at different velocities.

To select the velocities coressponding to the most "powerful" resonances (less suppressed by the dissipation), one must have in mind the following two rules: 1) the lower the resonance frequency, the more powerful the resonance; 2) the higher the load velocity, the more powerful the resonance. The first rule reflects the fact that the dissipation increases with the oscillation frequency; the second rule follows from the increase in power of transition radiation observed when the load velocity increases.

In conclusion, it should also be noted that the resonance conditions (17), though derived from our simple model, are general and permit the resonance parameters to be estimated for more complex models of the wheel that are closer to reality. According to (17), this problem is reduced to either analytical derivation or experimental determination of the dispersive dependence for waves propagating in the wheel.

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